Int. J. Solids Structures, Vol. 11, pp. 493–500. Pergamon Press 1975. Printed in Great Britain

# A NUMERICAL SOLUTION FOR AXIALLY SYMMETRICAL ELASTICITY PROBLEMS

THEODOR KERMANIDIS
Institut für Technische Mechanik, Technische Hochschule, Aachen, F.R.G.

(*Received* 27 *August* 1974)

**Abstract**—An integral equation method is presented for the solution of axially symmetrical elasticity problems. The obtained integral equations are of second kind with regular (Fredholm) and singular kernel. The method is suited to the treament of both simply and multiply connected regions with irregular boundary shapes and any boundary load distribution which satisfies the equilibrium conditions. Numerical results are included.

**Zusammenfassung**—Zur Lösung axialsymmetrischer Probleme der Elastizitätstheorie ist eine Integralgleichungsmethode gegeben. Die gewonnen Gleichungen sind 2. Art mit regulärem (Fredholm) und singulärem Kern. Die Methode eignet sich insbesondere zur Behandlung einfacher sowie mehrfach zusammenhängender Bereiche mit unregelmäßigen Grenzformen unter beliebiger Randbelastung, die lediglich die Gleichgewichtsbedingungen nicht verletzt. Numerische Ergebnisse werden gegeben.

## NOTATION

| | |
|---|---|
| $e, \vartheta, z$ | cylindrical coordinates |
| $\rho, \varphi, \zeta$ | cylindrical coordinates of a circular source |
| $n, \varphi, t$ | natural coordinates |
| $s(\bar{s})$ | boundary coordinate |
| $u, v, w$ | displacements corresponding to $e$, $\vartheta$, $z$ |
| $\sigma_{nn}, \sigma_{tt}, \sigma_{\vartheta\vartheta}$ | normal stress components |
| $\sigma_{nt}, \sigma_{n\vartheta}, \sigma_{t\vartheta}$ | shear stress components |
| $\nu$ | Poisson's ratio |
| $G$ | shear modulus |
| $\Omega$ | surface of the body |
| $B$ | meridian plane |
| $K^0$ | basic body |
| $K^+$ | real body |
| $K^-$ | difference body |
| $K((\pi/2), \kappa)$ | complete elliptic integral of the first kind |
| $E((\pi/2), \kappa)$ | complete elliptic integral of the second kind |
| $\kappa(\kappa')$ | modulus of the elliptic integral |
| $\mathbf{A}$ | coefficient matrix |
| $\Pi(s)$ | boundary load |
| $\mathbf{f}$ | boundary load vector |
| $R(s)$ | fictitious load |
| $\mathbf{q}$ | fictitius load vector |

## 1. INTRODUCTION

All known methods, exact or approximate, for the solution of axially symmetrical problems depend on the particular shape of the cross-section. This paper presents a numerical method for the axially symmetrical problem, which may be applied to any type of meridiam plane of the body of revolution. It is a singularity method using the principle of superposition of singularities distributed on the boundary of the solid, as Rieder[1], Massonnet[2], Kupradze[3], Weinel[9].

We have adopted as fundamental singularity the displacement field of a concentrated force in an elastic medium, which is used to produce any elastic field by superposition, as for example the displacement field of uniformly distributed forces on a circular shape. We note: (a) a radially loaded circular shape source, if the distributed forces are radial (Fig. 1a), (b) an axially loaded

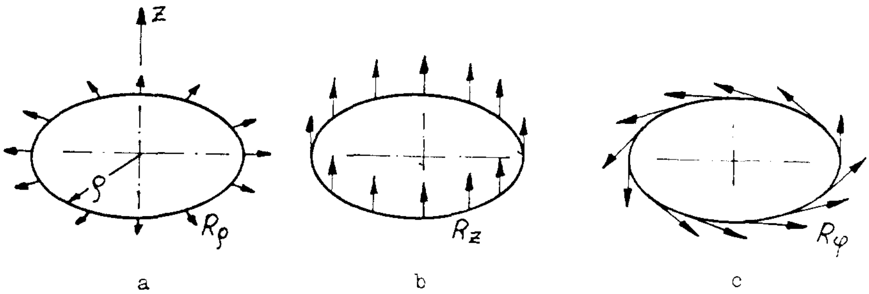


Fig. 1. Fundamental singularities; (a) radially, (b) axially, (c) tangentially loaded circular shape source.

circular shape source, if the distributed forces are axial (Fig. 1b), (c) a tangentially loaded circular shape source, if the distributed forces are tangential (Fig. 1c).

By the displacement field of these singularities the axially symmetrical problem can be reduced to a system of linear integral equations of second kind[1]. For this purpose we imagine the body of revolution with surface $\Omega$ inbedded in an elastic medium $K^0$, which is chosen to be homogeneous and infinite and to have the same elastic constants as the real body $K^+$. On the surface $\Omega$ in $K^0$ an axially symmetrical load distribution (fictitious load) is applied in such a manner that the stress- and displacement-field in the interior $B$ is the same as in $K^+$. We obtain a system of linear integral equations, a scalar integral equation of second kind (Fredholm integral equation), and a two component linear vectorial integral equation of second kind with singular kernel. The Fredholm integral equation describes the torsion problem of a body of revolution[3], and the vectorial singular integral equation describe the axially symmetrical problem for the deformation of the mid-section.

The integral equations will be solved numerically by dividing the contour $S$ of the meridian-shape (rotating about the $z$-axis $S$ is generating $\Omega$) into a finite number of segments $\Delta S$ and by replacing the contour integral by a sum extended over all segments. In this way we obain a system of simultaneous linear equations which can be solved on a computer by inversion of the matrix, or by successive approximation.

This method is especially convenient for axially symmetric problems with any shape of the mid-section and load distribution on the boundary, which has to be in equilibrium.

## 2. CHOICE OF THE FUNDAMENTAL SINGULAR DISPLACEMENT FIELD

In an infinite elastic medium $K^0$ the displacements $\mathbf{u}_i(x)$ of a concentrated force $F_i$ is given in the form of Papkowitsch-Neuber[2] by

$$\mathbf{u}(x) = \frac{\mathbf{F}}{4\pi GR} - \frac{1}{4(1-\nu)}\nabla\left(\mathbf{r}\cdot\frac{\mathbf{F}}{4\pi GR}\right) \tag{2.1}$$

or by

$$\mathbf{u}_i(x) = \frac{1}{16\pi G(1-\nu)}\left[(3-4\nu)\frac{\mathbf{F}_i}{R} + \frac{(x_i-\zeta_i)(x_j-\zeta_j)}{R^3}\mathbf{F}_j\right] \tag{2.2}$$

where $\mathbf{r}$ is the radius vector from the origin $\zeta_i$ of the force $\mathbf{F}_i$ to the point $x_i$ with the coordinates $x$, $y$, $z$, $R$ is the value of $\mathbf{r}$, $\nu$ the Poisson's ratio and $G$ the shear modulus. $(\cdot)$ is the scalar product. By superposition of (2.1) or (2.2) any elastic displacement field can be produced in $K^0$. We obtain the displacement field of a loaded circular shape source with the help of Betti's method. The interaction energy of two elastic stress states is given by Betti as

$$A_{p,u} = \int_{(S_p)} \mathbf{F}(x_i)\mathbf{u}(\zeta_i, x_i)\,\mathrm{d}S \qquad x, \zeta \epsilon S_p. \tag{2.3}$$

$A_{p,u}$ is the work of the loading $\mathbf{F}_i$ to the displacement field $u$ of a single force $\mathbf{F}_0$ acting at the point $\zeta_i$; $S_p$ is the surface on which $\mathbf{F}_i$ is applied.

(a) *Displacement field of a radially loaded circular shape source.* The displacement field of a radially loaded circular shape source with the intensity $2\pi R_\rho \rho$ is obtained by equation (2.3), if we deal with the loading $\mathbf{F}$ a uniformly radial load distributed over the circumference of the shape with the density $R_\rho$ and with the single force $\mathbf{F}_0$ at $\zeta_i$ a unit force in radial($e$), axial($z$) and tangential($\vartheta$) directions. Then equation (2.3) gives the displacement vector in cylindrical coordinates (Fig. 2)

$$u = \frac{R_\rho}{4\pi G(1-\nu)}\left[\frac{4(1-\nu)(e^2+\rho^2+\bar{z}^2)-(e^2+\rho^2)}{2e\sqrt{((e+\rho)^2-\bar{z}^2)}}K\left(\frac{\pi}{2},\kappa\right)+\right.$$
$$\left.-\left(\frac{3\cdot 5-4\nu}{2e}\sqrt{((e+\rho)^2+\bar{z}^2)}-\frac{(e^2-\rho^2)^2-\bar{z}^4}{4e\sqrt{((e+\rho)^2+\bar{z}^2)^3}(1-\kappa^2)}\right)E\left(\frac{\pi}{2}\kappa\right)\right]$$

$$v = 0 \tag{2.4}$$

$$w = \frac{R_\rho \bar{z}}{4\pi G(1-\nu)}\left[\frac{(e^2-\rho^2+\bar{z}^2)E\left(\frac{\pi}{2},\kappa\right)}{2\sqrt{((e+\rho)^2+\bar{z}^2)^3}(1-\kappa^2)}-\frac{K\left(\frac{\pi}{2},\kappa\right)}{2\sqrt{((e+\rho)^2+\bar{z}^2)}}\right].$$

$K((\pi/2), \kappa)$ and $E((\pi/2), \kappa)$ are the complete elliptic integrals of the first and second kind, with

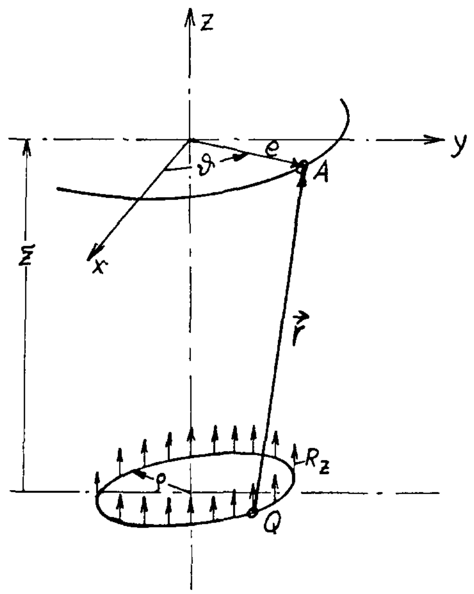


Fig. 2. Geometry corresponding to the displacement field of the axial loaded circular shape source.

the modulus $\kappa$, $(\tilde{z} = z - \zeta)$

$$\kappa^2 = \frac{4e\rho}{(e+\rho)^2 + \tilde{z}^2}; \qquad 0 \leq \kappa^2 \leq 1 \tag{2.5}$$

and the complementary modulus of $\kappa$

$$\kappa' = \sqrt{(1-\kappa^2)}. \tag{2.6}$$

This displacement field is independent from the angle $\vartheta$.

(c) *Displacement field of an axially loaded circular shape source.* In analogy to the displacement field of equation (2.4) the equation for the axially loaded circular shape source with the intensity $2\pi R_z\rho$ follows

$$u = \frac{R_z\rho\tilde{z}}{4\pi G(1-\nu)}\left[\frac{(e^2-\rho^2-\tilde{z}^2)}{2e\sqrt{((e+\rho)^2+\tilde{z}^2)^3}(1-\kappa^2)}E\left(\frac{\pi}{2},\kappa\right) + \frac{K((\pi/2),\kappa)}{2e\sqrt{((e+\rho)^2+\tilde{z}^2)}}\right]$$

$$v = 0 \tag{2.7}$$

$$w = \frac{R_z\rho}{4\pi G(1-\nu)}\left[\frac{3-4\nu}{\sqrt{((e+\rho)^2+\tilde{z}^2)}}K\left(\frac{\pi}{2},\kappa\right) + \frac{\tilde{z}^2E((\pi/2),\kappa)}{\sqrt{((e+\rho)^2+\tilde{z}^2)^3}(1-\kappa^2)}\right].$$

For $\rho \to 0$ and $2\pi R_z\rho$ = konst. the displacement field (2.7) is tending to the displacement field of a concentrated force in $z$-direction.

(c) *Displacement field of a tengentially loaded circular shape source.* The displacement field of this fundamental singularity with intensity $2\pi R_\varphi\rho^2$ is given by

$$u = w = 0 \tag{2.8}$$

$$v = -\frac{R_\varphi}{2\pi Ge}\left[\sqrt{((e+\rho)^2+\tilde{z}^2)}E\left(\frac{\pi}{2},\kappa\right) - \frac{(e^2+\rho^2+\tilde{z}^2}{\sqrt{((e+\rho)^2+\tilde{z}^2)}}K\left(\frac{\pi}{2},\kappa\right)\right].$$

From equations (2.4), (2.7), (2.8) follows that the radial and longitudinal displacements $u$ and $w$ depend only on the radially and on the axially loaded circular shape source, whereas the cross-radial displacement $v$ depends only on the tangentially loaded circular shape source.

2.1 *The Stress field of the fundamental singularities*

The displacements in an infinite elastic medium $K^0$ loaded by a circular shape source are given in equations (2.4), (2.7), (2.8) respectively. The stress components can then be obtained from Lame's equations [4]

$$\sigma_{\rho\rho} = \frac{2G\nu}{1-2\nu}\left[\frac{u}{e} + \frac{\partial w}{\partial z} + \left(\frac{1-\nu}{\nu}\right)\frac{\partial u}{\partial e}\right]$$

$$\sigma_{zz} = \frac{2G\nu}{1-2\nu}\left[\frac{u}{e} + \frac{\partial u}{\partial e} + \left(\frac{1-\nu}{\nu}\right)\frac{\partial w}{\partial z}\right]$$

$$\sigma_{\vartheta\vartheta} = \frac{2G\nu}{1-2\nu}\left[\frac{\partial u}{\partial e} + \frac{\partial w}{\partial z} + \left(\frac{1-\nu}{\nu}\right)\frac{u}{e}\right]$$

$$\sigma_{\rho z} = G\left(\frac{\partial w}{\partial e} + \frac{\partial u}{\partial z}\right)$$

$$\sigma_{\vartheta z} = G\frac{\partial v}{\partial z}$$

$$\sigma_{\rho\vartheta} = G\left(\frac{\partial v}{\partial e} - \frac{u}{e}\right) \tag{2.9}$$

with the partial derivatives of the elliptic functional equations of the argument $x = \kappa^2$.

$$\frac{\mathrm{d}E\left(\frac{\pi}{2},\kappa\right)}{\mathrm{d}x} = \frac{E\left(\frac{\pi}{2},\kappa\right) - K\left(\frac{\pi}{2},\kappa\right)}{2x}$$

$$\frac{\mathrm{d}K\left(\frac{\pi}{2},\kappa\right)}{\mathrm{d}x} = \frac{E\left(\frac{\pi}{2},\kappa\right)}{2x(1-x)} - \frac{K\left(\frac{\pi}{2},\kappa\right)}{2x}. \tag{2.10}$$

$G$ is the shear modulus; $\nu$ is the Poisson's ratio. Solving directly for displacements, compatibility of strains is identically satisfied.

## 3. FORMULATION OF THE INTEGRAL EQATION OF THE BOUNDARY VALUE PROBLEM

The boundary $\Omega$ of the body of revolution will be assumed to be smooth. It may be loaded by a symmetrically load distribution $\Pi(s)$ which satisfies the equilibrium conditions and without discontinuities can be arbitrary on the boundary $s$.

For the solution of this boundary value problem we will formulate a system of linear integral equations, which can be solved numerically. For this purpose we imagine the body of revolution "the real body $K^+$" inbedded in an elastic medium, "the basic body $K^0$". The homogeneous body $K^0$ is chosen to be infinite and to have the same elastic constants as the real body $K^+$. A fictitious load $R(s)$ will be applied on the surface $\Omega$ in $K^0$, which is assumed to be symmetrical with respect to the $z$-axis. This procedure is characteristic for Rieder's method in plane elasticity [1]. The stress and displacement field in $K^0$ can be produced by superposition of the fundamental singularities which are given above. For a point $A$ lying inside or outside of $\Omega$ the stress components are given by

$$\sigma_{nn} = \int_S K_{11}(s,\bar{s})R_n(\bar{s})\,\mathrm{d}\bar{s} + \int_S K_{13}(s,\bar{s})R_t(\bar{s})\,\mathrm{d}\bar{s}$$

$$\sigma_{nt} = \int_S K_{23}(s,\bar{s})R_n(\bar{s})\,\mathrm{d}\bar{s} + \int_S K_{32}(s,\bar{s})R_t(\bar{s})\,\mathrm{d}\bar{s}$$

$$\sigma_{tt} = \int_S K_{31}(s,\bar{s})R_n(\bar{s})\,\mathrm{d}\bar{s} + \int_S K_{33}(s,\bar{s})R_t(\bar{s})\,\mathrm{d}\bar{s}$$

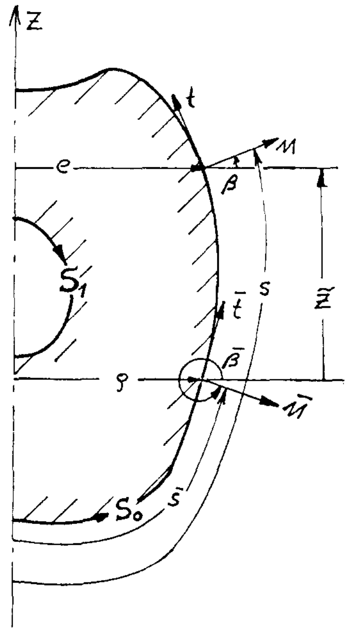


Fig. 3. To the equations (3.1), (3.2) and (3.5).

$$\sigma_{\vartheta\vartheta} = \int_S K_{12}(s, \bar{s}) R_n(\bar{s})\, \mathrm{d}\bar{s} + \int_S K_{21}(s, \bar{s}) R_t(\bar{s})\, \mathrm{d}s \tag{3.1}$$

$$\sigma_{n\vartheta} = \int_S K_{22}(s, \bar{s}) R_\varphi(\bar{s})\, \mathrm{d}\bar{s}$$

$$\sigma_{t\vartheta} = \int_S K^*_{23}(s, \bar{s}) R_\varphi(\bar{s})\, \mathrm{d}\bar{s}.$$

The line integral is taken around the boundary $S$ (Fig. 3) in the $\rho z$-plane and the boundary loading

$$R_n = R_\rho \cos\bar{\beta} + R_z \sin\bar{\beta}; \qquad R_t = -R_\rho \sin\bar{\beta} + R_z \cos\bar{\beta} \tag{3.2}$$

and the stress components corresponding to $n, \varphi, t$

$$\sigma_{nn} = \sigma_{\rho\rho} \cos^2\bar{\beta} + \sigma_{\rho z} \sin 2\bar{\beta} + \sigma_{zz} \sin^2\bar{\beta}$$

$$\sigma_{nt} = -\frac{1}{2}\sigma_{\rho\rho} \sin 2\bar{\beta} + \sigma_{\rho z} \cos 2\bar{\beta} + \frac{1}{2}\sigma_{zz} \sin 2\bar{\beta} \tag{3.3}$$

$$\sigma_{tt} = \sigma_{\rho\rho} \sin^2\bar{\beta} - \sigma_{\rho z} \sin 2\bar{\beta} + \sigma_{zz} \cos^2\bar{\beta}$$

where $\bar{\beta}$ is the angle between the normal $\bar{n}$ and the $\rho$-axis. The kernels $K_{11}, K_{12}, \ldots$ can easily be determined by equation (2.9). If the point $A$ with the coordinates $x_i$ converges to a point $Q$ on the boundary the stress vector presents a discontinuity equal

$$\Delta\sigma_{nn}(s) = \frac{1}{2}R_n(s); \qquad \Delta\sigma_{nt}(s) = \frac{1}{2}R_t(s)$$

$$\Delta\sigma_{tt}(s) = \Delta\sigma_{\vartheta\vartheta}(s) = \frac{\nu}{2(1-\nu)}R_n(s); \qquad \Delta\sigma_{n\vartheta}(s) = \frac{1}{2}R_\varphi(s) \tag{3.4}$$

This discontinuity is independent of the way in which the point $A$ is approaching the boundary[6]. The stress components on the boundary element are therefore given by the expressions (3.1) and (3.4).

If we desire to determine the tensor field in the interior of an elastic body of revolution subjected on its edge to given surface forces $\Pi(s)$ we will have to choose the following distribution of fictitious loads $R(s)$:

$$\sigma_{nn}(s) = \Pi_n(s); \qquad \sigma_{nt}(s) = \Pi_t(s); \qquad \sigma_{n\vartheta}(s) = \Pi_\varphi(s) \tag{3.5}$$

Thus at any point of the boundary the stress equilibrates exactly the surface force $\Pi(s)$. That gives $\oint$ meaning Cauchy principal value, (see also[9])

$$\begin{aligned}
&\frac{1}{2}R_n(s) + \oint_{(S,\, s\neq\bar{s})} K_{11}(s,\bar{s})R_n(\bar{s}\,\mathrm{d}\bar{s} + \oint_{(S,\bar{s}\neq s)} K_{13}(s,\bar{s})R_t(\bar{s})\,\mathrm{d}\bar{s} = \Pi_n(s)\\
&\frac{1}{2}R_t(s) + \oint_{(S,\,\bar{s}\neq s)} K_{23}(s,\bar{s})R_n(\bar{s})\,\mathrm{d}\bar{s} + \oint_{(S,\,\bar{s}\neq s)} K_{32}(x,\bar{s})R_t(\bar{s})\,\mathrm{d}\bar{s} = \Pi_t(s)\\
&\frac{1}{2}R_\varphi(s) + \lim_{\bar{s}\to s}\int_{(S)} K_{22}(s,\bar{s})R_\varphi(\bar{s})\,\mathrm{d}\bar{s} = \Pi_\varphi(s).
\end{aligned} \tag{3.6}$$

This is a linear vectorial integral equation of the second kind. The first two components are singular integral equations and the third a Fredholm one. This scalar integral equation of Fredholm type describes the torsional problem of an elastic body of revolution for any load distribution on the boundary, which is in equilibrium. The two singular integral equations describe the axially symmetrical problem on which the mid-section plane will be deformated. This system of the integral equations can only be solved numerically.

## 4. NUMERICAL SOLUTION

For the numerical solution of this system the contour $S$ of the meridian plane $(ez)$ (Fig. 4) will be approximated by $N$ straight or circular segments with tangential transmission. The fictitious load distribution over $S$ can be approximated linearly between the knots $j$ and $j+1$ ($j = 1, 2, \ldots N+1$) or by other functions, as, for example spline functions[8]. By these finite methods the vectorial integral equation is reduced to a system of linear algebraic equations

$$\mathbf{Aq} = \mathbf{f}. \tag{4.1}$$

$\mathbf{A}$ is a quadratic generally not symmetric matrix; $\mathbf{f}$ rsp., $\mathbf{q}$ are the vectors of the given rsp. fictitious load distribution on $S$ with the components $\Pi_j$ rsp. $R_j$ at the knots.

As known, integral methods yield systems of equations with much less unknowns than finite difference or finite elements techniques. On the other hand, the coefficient matrix of this system is usually complete, so that one cannot benefit from special devices applicable to the solution of band or sparse matrices.

This equation was here solved by the Gauss inversion method. Now let us demonstrate briefly how this method may be used practically for solving the torsion problem of a cylinder with a

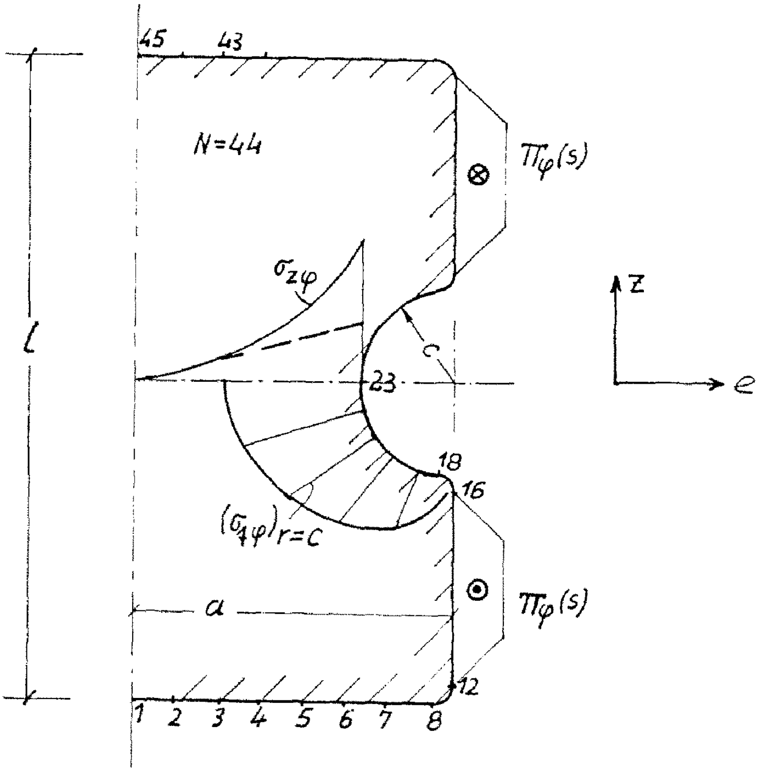


Fig. 4. Cylinder with a circular notch. $\sigma_{t\varphi}$, $\sigma_{z\varphi}$ are the stresses calculated by the integral equation method,--- is the shearing stress by $(M_T/I_P)e$.

circular notch (Fig. 4). We take a cylinder with radius $a$, length $l = 2a$, and radius of the circular notch $c = a/3$. For $N = 44$ segments on $S$ equation (3.6.3) is approximated by

$$\frac{1}{2}R_{\varphi}(s_j)+\sum_{i=1}^{N}\int_{s_i}^{s_{i+1}} K_{22}(s_j,\bar{s})\left[R_{\varphi}(s_i)\frac{s_{i+1}-\bar{s}}{s_{i+1}-s_i}+R_{\varphi}(s_{i+1})\frac{\bar{s}-s_i}{s_{i+1}-s_i}\right]\mathrm{d}\bar{s}=\Pi_{\varphi}(s_j). \tag{4.2}$$

These $N$ linear algebraic equation gives the approximate fictitious load distribution, by which we can compute with the expressions (3.1) the stresses $\sigma_{t\varphi}$ on the boundary of the notch and $\sigma_{z\varphi}$ in the interior, the results are presented in Fig. 4.

## REFERENCES

1. G. Rieder Iterationsverfahren und Operatorgleichungen in der Elastizitätstheorie. *Abh. der Braunschw. Wiss. Ges.* **14**, 109 (1962).
2. Ch. Massonnet, Solution générale du problème aux tensions de l'élasticité tridimensionelle. *Proc. 9th Intern. Congr. Appl. Mech.* S. 168–180, Bruxelles (1956).
3. V. D. Kupradze, T. G. Geglia, M. O. Baselejsvili and T. B. Burculadze, Trechmernye zadaci matematiceskoj teorii yprogosti. Russisch (Thilisi 1968).
4. L. N. G. Filon, On the Elastic Equilibrium of Circular Cylinders under Certain Practical Systems of Load. Phil. Transaction of the Royal society of London, A 198 (1902) S. 147.
5. Th. Kermanidis, Eine Integralgleichungsmethode zur Lösung des Torsionsproblems des Umdrehungskörpers. *Acta Mechanica* **16**, 175 (1973).
6. Th. Kermanidis, Eine Integralgleichungsmethode zur Lösung der rotationssymmetrischen Randwertprobleme der Elastizitätstheorie, insbesondere des Torsionsproblems. Dissertation Aachen.
7. U. Heise, Über die Spektren einiger Integraloperatoren für die isotrope elastische Kreisscheibe. *ZAMM* **50**, T122 (1970).
8. H. Antes, Splinefunktionen bei der Lösung von Integralgleichungen. *Numer. math.* **19**, S. 116–126 (1972).
9. E. Weinel, Über achsensymmetrische Randwertaufgaben der Elastizitätstheorie. *ZAMM* **11**, 438 (1931).